urement of the flow conditions at the exit section. The determination of the effects of shock/boundary-layer interaction in the nozzle on the shock shape and flow conditions at the exit section are also needed.

Extension of the present approach to the case of reacting fluids would again lead to a method of analyzing thrust vector control without seeking many of the details of a much more complicated flow field. (This extension is now under study.) The authors expect that, in the case of the reacting fluids, just as in the present case of nonreacting fluids, the problem would reduce to the determination of a few variables instead of that of all the variables. The solution for these may then be attempted on the basis of some over-all considerations or by suitably combining theoretical and experimental results.

By analyzing thrust vector control on the basis of the present approach, one may be able to pick out the important nondimensional parameters in the problem and know, to a satisfactory extent, the functional dependence of the side force on those parameters. For the case of inert gases, it can be shown, on the basis of Eqs. (14, 18, and 19), that the nondimensional side force is given by a relation of the form

$$\frac{F_s}{\dot{m}_0 U_e} = f\left(\frac{\dot{m}_i}{\dot{m}_0}, \frac{H_i}{U_0^2}, \frac{H_0}{U_e^2}, \gamma_0, \frac{\gamma_i}{\gamma_0}, \frac{\mu_i}{\mu_0}, \frac{p_e A_e}{\dot{m}_0 U_e}, \frac{\epsilon_i}{\epsilon}, \frac{l}{d}\right)$$

where γ denotes specific heat ratio, and μ denotes molecular weight.

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Optimal Programming Problems with Inequality Constraints I: Necessary Conditions for Extremal Solutions

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The necessary conditions are presented for an extremal solution to a programming problem with an inequality constraint on a function of the control and/or the state variables. It is shown that, in general, certain terms must be added to the Euler-Lagrange equations during intervals in which the solution curve lies on the boundary. Furthermore, for an inequality constraint function *not* explicitly involving the control variable(s), one or more functions of the state and time must satisfy equality constraints at the beginning (the entry corner) of an inequality constraint boundary interval. These constraints cause discontinuities in the influence functions (Lagrange multiplier functions) at the entry corner. The derivation of the necessary conditions which is given may also be used to allow the equations of motion to be discontinuous or even integrably infinite functions of the state as well as the time at a finite number of points. Two analytic example problems with state variable inequality constraints are presented.

1. Introduction

IN the calculus of variations, the problem of Bolza (the Mayer formulation is used) has been and continues to be of major significance. A dynamical system is considered which is represented at any time by the values of its state variables and whose development in time is determined by choices of control variable program(s). The Bolza problem asks for that control variable program(s) which will maximize (minimize) a given function of the state, while constraining other functions to specified values, at the terminal point. In this paper, the primary concerns are the modifications and additions to the necessary conditions for an extremal solution of the Bolza problem when there is an inequality constraint imposed, along the entire path, upon some function of the control and/or state variables.

Problems involving inequality constraint(s) on the control variable(s) were treated as early as 1937 by Valentine.[1] More recently, they were discussed by Cicala[2] and by Breakwell.[3] Problems involving inequality constraints on a function of the state variables with no explicit dependence on the control variables have been treated only in recent years. Gamkrelidze[4] in 1960 presented necessary conditions for extremal solutions assuming that the time derivative of the inequality constraint function was an explicit function of the control variable(s). Berkovitz[5] obtained essentially equiva-

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lent results in a supplement to an earlier paper of his. In both cases, it was demonstrated that there is a certain nonuniqueness of the discontinuities in the influence functions (also called adjoint variables or Lagrange multiplier functions) at the corners of entry onto and exit from the constraint boundary. Although this is mathematically correct, the approach used here makes a specific choice in the manner naturally consistent with the other boundary conditions.

Dreyfus[6] considered the more general state variable inequality constraint where a time derivative higher than the first might be required to involve the control variable(s) explicitly. If the qth derivative of the inequality constraint function is the first to be an explicit function of the control variable(s), the constraint is called a qth order state variable inequality constraint. On such a constraint boundary, one control variable is determined by the requirement that the qth derivative of the constraint function be zero. Furthermore, all lower derivatives must be zero at the entry corner (onto the constraint boundary) in order that the constraint not be violated. These entry corner constraints are always equal in number to the number of relations needed to determine the jumps in the adjoint variables across the entry corner. In Ref. 6, Dreyfus gave equations for determining the jumps in the case of the first-order inequality constraint. The results in Ref. 6 involved a reduced set of adjoint variables on the constraint boundary rather than a full set of such variables with jumps; consequently, they cannot be related simply to the results of Gamkrelidze[4] and Berkovitz.[5]

In this paper, the necessary conditions are derived, including corner conditions, for an extremal solution to the general qth order inequality constraint problem. The classical approach of adjoining the constraints to the performance index by Lagrange multipliers is used. It is possible to use the derivation to relax somewhat the usual continuity requirements on the dynamical system differential equations.

Two analytical examples are presented to illustrate the use of the theoretical results.

2. Optimal Programming Problem with an Inequality Constraint

The Mayer formulation of the problem of Bolza is the determination of $\alpha(t)$ in the interval $t_0 \leq t \leq t_f$ so as to maximize

$$J = \phi[\mathbf{x}(t_f),t_f] \tag{2.1}$$

subject to the constraints

$$\dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}(t),\alpha(t),t] \tag{2.2}$$

$$\mathbf{M} = \mathbf{M}[\mathbf{x}(t_f),t_f] = 0 \tag{2.3}$$

$$t_0 \text{ and } \mathbf{x}(t_0) \text{ given} \tag{2.4}$$

where t is the independent variable, hereafter called time; $(\dot{\ })$ is $d/dt(\ )$; $\alpha(t)$ is a scalar§ control variable that is freely chosen;

$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}$ is an n-vector of state variable histories, which result from given values $\mathbf{x}(t_0)$ and a choice of $\alpha(t)$

$\mathbf{f} = \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}$ is an n-vector of known functions of $\mathbf{x}(t),\alpha(t)$, and t, assumed everywhere differentiable with respect to $\mathbf{x}$ and α

§ Problems with more than one constraint and/or more than one control variable are treated in Ref. 7.

ϕ is the performance index and is a known function of $\mathbf{x}(t_f)$ and t_f;

$\mathbf{M} = \begin{bmatrix} M_1 \\ \cdot \\ \cdot \\ \cdot \\ M_p \end{bmatrix}$ is a p-vector of terminal constraint functions, each of which is a known function of $\mathbf{x}(t_f)$ and t_f; p must be $\leq n$

Of primary interest is the addition of an inequality constraint

$$C(\mathbf{x},\alpha,t) \leq 0 \tag{2.5}$$

or

$$S(\mathbf{x},t) \leq 0 \tag{2.6}$$

where C is a scalar function of $\mathbf{x}(t),\alpha(t)$ and t; S is a scalar function of $\mathbf{x}(t)$ and t‖; and $\alpha(t)$ must remain within the limits imposed by $C \leq 0$ or $S \leq 0$. In the derivation in Appendix A of the necessary conditions for an extremal solution with $S \leq 0$, other extensions to the classical problem of Bolza are included.

3. Necessary Conditions for an Extremal Solution with a Control Variable Inequality Constraint

Considered first is the problem with a constraint relation that explicitly involves the control variable program, $C(\mathbf{x},\alpha,t) \leq 0$. The constraint function may also involve the state variables $\mathbf{x}(t)$ and/or be an explicit function of the independent variable t.

For those periods of an extremal solution on the constraint boundary, $\alpha(t)$ is determined in terms of the state variables and the independent variable by the relation

$$C[\mathbf{x}(t),\alpha(t),t] = 0 \tag{3.1}$$

Thus, the neighboring solutions for those periods must satisfy

$$(\partial C/\partial \mathbf{x})\delta\mathbf{x} + (\partial C/\partial\alpha)\delta\alpha = 0 \tag{3.2}$$

where $\partial C/\partial \mathbf{x} = [\partial C/\partial x_1, \ldots, \partial C/\partial x_n]$ and $\partial C/\partial\alpha$ are evaluated along the extremal solution. Neighboring solutions must also satisfy the perturbation differential equations

$$\frac{d}{dt}(\delta\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\delta\mathbf{x} + \frac{\partial \mathbf{f}}{\partial\alpha}\delta\alpha \tag{3.3}$$

where

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad \frac{\partial \mathbf{f}}{\partial\alpha} = \begin{bmatrix} \frac{\partial f_1}{\partial\alpha} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f_n}{\partial\alpha} \end{bmatrix}$$

are evaluated along the extremal solution. Substituting (3.2) into (3.3) yields

$$\frac{d}{dt}(\delta\mathbf{x}) = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \frac{\partial \mathbf{f}}{\partial\alpha}\left(\frac{\partial C}{\partial\alpha}\right)^{-1}\frac{\partial C}{\partial \mathbf{x}}\right]\delta\mathbf{x} \tag{3.4}$$

which is the set of perturbation equations which a neighboring solution must satisfy if it is to remain on the constraint boundary $C = 0$. It follows that the Euler-Lagrange equations for determining an extremal solution are

$$\frac{d\lambda}{dt} = \begin{cases} -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^T \lambda & \text{when } C < 0 \\ -\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \frac{\partial \mathbf{f}}{\partial\alpha}\left(\frac{\partial C}{\partial\alpha}\right)^{-1}\frac{\partial C}{\partial \mathbf{x}}\right]^T \lambda & \text{when on the constraint boundary } C = 0 \end{cases} \tag{3.5}$$

‖ See previous footnote.

$$\lambda^T \frac{\partial \mathbf{f}}{\partial \alpha} = 0 \text{ determines } \alpha(t) \text{ when } C < 0 \tag{3.6}$$

$$C(\mathbf{x},\alpha,t) = 0 \text{ determines } \alpha(t) \text{ when on the constraint boundary} \tag{3.7}$$

where

$$\lambda(t) = \begin{bmatrix} \lambda_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \lambda_n(t) \end{bmatrix}$$

is an n-vector of influence functions, the Lagrange multiplier functions, and where $(\ )^T$ is the transpose of $(\ )$.

In an interval of $C = 0$, the inequality

$$\frac{\lambda^T(\partial \mathbf{f}/\partial \alpha)}{\partial C/\partial \alpha} > 0 \tag{3.8}$$

must be satisfied. This is equivalent to the requirement that the variational Hamiltonian be minimized,# with respect to permissible α, by the α from (3.7). It means that the only perturbations in α which could increase the performance index ϕ would violate $C \leq 0$. If this were not the case, then (3.6) and the first member of (3.5) would be used. The boundary conditions for the influence functions are

$$\lambda^T(t_f) = \left[\frac{\partial \phi}{\partial \mathbf{x}} + \boldsymbol{\nu}^T \frac{\partial \mathbf{M}}{\partial \mathbf{x}}\right]_{t=t_f} \tag{3.9}$$

with the transversality condition

$$(\lambda^T \dot{\mathbf{x}})_{t=t_f} = -\left[\frac{\partial \phi}{\partial t} + \boldsymbol{\nu}^T \frac{\partial \mathbf{M}}{\partial t}\right]_{t=t_f} \tag{3.10}$$

where $\boldsymbol{\nu}$ is a p-vector of Lagrange multiplier constants. The initial time and initial state are given. The influence function boundary values (at the terminal time), the terminal time itself, and the constants $\boldsymbol{\nu}$ constitute $n + 1 + p$ quantities, which are determined so as to satisfy (3.9), (3.10), and the constraint relations (2.3).

The control variable α *may* have finite discontinuities at the entering or exit corners (and even at other points), but this only produces discontinuities in $d\lambda/dt$, not λ. Thus, the influence functions λ are continuous across such corners; in addition, the Hamiltonian $\lambda^T \dot{\mathbf{x}}$ is continuous across corners (although some components of $\dot{\mathbf{x}}$ must be discontinuous if α is discontinuous). These are the Erdmann corner conditions in modern notation.

4. Necessary Conditions for an Extremal Solution with a State Variable Inequality Constraint

Considered here is the problem with a constraint function that is an explicit function of the state variables and the independent variable only, $S(\mathbf{x},t) \leq 0$. For those periods of an

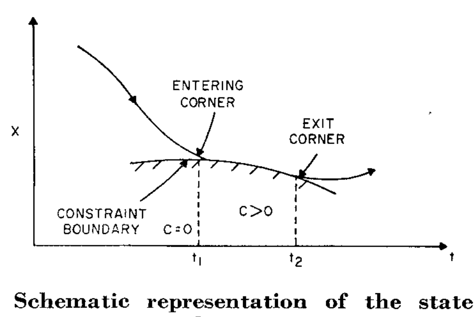

Fig. 1 Schematic representation of the state vector history.

In the book listed under Ref. 4, the maximum principle states that the Hamiltonian is maximum at each point for a minimum of J.

extremal solution on the constraint boundary, the state variables are interrelated by

$$S[\mathbf{x}(t),t] = 0 \tag{4.1}$$

Since the constraint function S must vanish identically when the solution lies on the constraint boundary, it follows that its time derivatives also must vanish:

$$d^kS/dt^k = S^{(k)} = 0$$
$$k = 0,1,2, \ldots \text{ on the constraint boundary} \tag{4.2}$$

Now

$$\begin{aligned} dS/dt &= (\partial S/\partial t) + (\partial S/\partial \mathbf{x})\dot{\mathbf{x}} \\ &= (\partial S/\partial t) + (\partial S/\partial \mathbf{x})\mathbf{f} \end{aligned} \tag{4.3}$$

and, since $\mathbf{f}(\mathbf{x},\alpha,t)$ appears in this latter relation, dS/dt *may* be an explicit function of the control variable $\alpha(t)$. If it is *not* an explicit function of $\alpha(t)$, one may consider the second derivative, third derivative, etc., until one finally comes to a time derivative that does explicitly involve the control variable. Suppose this is the qth time derivative; this will be called a qth order state variable inequality constraint. $S^{(q)}[\mathbf{x},\alpha,t] = 0$ then plays exactly the same role as $C[\mathbf{x},\alpha,t] = 0$ in the previous section, i.e., it determines $\alpha(t)$ in terms of $\mathbf{x}(t)$ and t when the solution is on the constraint boundary $S = 0$. Thus, the differential equations for the influence functions $\lambda(t)$ are the same as in (3.5) with C replaced by $S^{(q)}$.

However, in addition to the fact that $S^{(q)}[\mathbf{x},\alpha,t] = 0$ on the constraint boundary, it must also be stipulated that at entering corners (points where the solution goes from an unconstrained arc onto a constrained arc; see Fig. 1), the following conditions be met:

$$\begin{aligned} S[\mathbf{x}(t_1),t_1] &= 0 \\ S^{(1)}[\mathbf{x}(t_1),t_1] &= 0 \\ &\ \ \vdots \\ S^{(q-1)}[\mathbf{x}(t_1),t_1] &= 0 \end{aligned} \tag{4.4}$$

Now, Eqs. (4.4) play the role of *terminal constraints* for the unconstrained arc just preceding the constrained arc. It follows immediately that the influence functions $\lambda(t)$ must satisfy relations similar to those in Eqs. (3.9) and (3.10) at the entering corner where $t = t_1$. These relations are

$$\lambda^T(t_1-) = \lambda^T(t_1+) + \left[\mu_0 \frac{\partial S}{\partial \mathbf{x}} + \mu_1 \frac{\partial S^{(1)}}{\partial \mathbf{x}} + \ldots + \mu_{q-1} \frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\right]_{t=t_1} \tag{4.5}$$

$$[\lambda^T \dot{\mathbf{x}}]_{t=t_1-} = [\lambda^T \dot{\mathbf{x}}]_{t=t_1+} - \left[\mu_0 \frac{\partial S}{\partial t} + \mu_1 \frac{\partial S^{(1)}}{\partial t} + \ldots + \mu_{q-1} \frac{\partial S^{(q-1)}}{\partial t}\right]_{t=t_1} \tag{4.6}$$

where $\mu_0, \ldots, \mu_{q-1}$ are q Lagrange multiplier constants. The influence function values at t_1-, the entry corner time t_1, and constants $\mu_0, \ldots, \mu_{q-1}$ constitute $n + 1 + q$ quantities that are determined so as to satisfy (4.5), (4.6), and the constraint relations (4.4). Thus, the influence functions are, in general, *discontinuous* across an entering corner. They are still continuous, however, at exit corners (points where the solution goes from a constrained arc onto an unconstrained arc; see Fig. 1):

$$\lambda(t_2-) = \lambda(t_2+) \tag{4.7}$$

Also,

$$(\lambda^T \dot{\mathbf{x}})_{t=t_2-} = (\lambda^T \dot{\mathbf{x}})_{t=t_2+} \tag{4.8}$$

Gamkrelidze[4] and Berkovitz[5] showed, for the first-order state variable inequality constraint, that in the relation

$$\lambda^T(t_1-) = \lambda^T(t_1+) + \mu_0(\partial S/\partial \mathbf{x})_{t=t_1} \quad (4.9)$$

an arbitrary constant, say b, could be added to μ_0. At the same time, they showed that (4.7) could be written

$$\lambda(t_2-) = \lambda(t_2+) + b'(\partial S/\partial \mathbf{x})_{t=t_2} \quad (4.10)$$

with an arbitrary constant b'. What was not stated in Ref. 4 or 5 is that b' *must* equal $(-b)$. In other words, the jump at t_1 is not unique, but the jump at t_2 is determined by the jump at t_1. Thus one has the pair

$$\begin{cases} \lambda^T(t_1-) = \lambda^T(t_1+) + (\mu_0 + b)(\partial S/\partial \mathbf{x})_{t=t_1} \\ \lambda^T(t_2-) = \lambda^T(t_2+) - b(\partial S/\partial \mathbf{x})_{t=t_2} \end{cases} \quad (4.11)$$

where b is arbitrary, but μ_0 is unique, for the first-order state variable inequality constraint. For the qth order constraint, the corresponding results are

$$\begin{aligned} \lambda(t_1-) &= \lambda(t_1+) + \left[\mu_0 \frac{\partial S}{\partial \mathbf{x}} + \mu_1 \frac{\partial S^{(1)}}{\partial \mathbf{x}} + \ldots + (\mu_{q-1} + b)\frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\right]_{t=t_1} \\ \lambda(t_2-) &= \lambda(t_2+) - b\left(\frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\right)_{t=t_2} \end{aligned} \quad (4.12)$$

The jumps in the variational Hamiltonian $\lambda^T\dot{\mathbf{x}}$ at the corners obey the analogous relations

$$\begin{aligned} (\lambda^T\dot{\mathbf{x}})_{t=t_1-} &= (\lambda^T)\dot{\mathbf{x}}_{t=t_1+} - \left[\mu_0 \frac{\partial S}{\partial t} + \ldots + (\mu_{q-1} + b)\frac{\partial S^{(q-1)}}{\partial t}\right]_{t=t_1} \\ (\lambda^T\dot{\mathbf{x}})_{t=t_2-} &= (\lambda^T\dot{\mathbf{x}})_{t=t_2+} + b\left(\frac{\partial S^{(q-1)}}{\partial t}\right)_{t=t_2} \end{aligned} \quad (4.13)$$

A demonstration of why the arbitrary b is allowed and why b' must equal $(-b)$ is given in Appendix B.

5. Analytical Example with a First-Order State Variable Inequality Constraint: A Brachistochrone Problem

Given

$$\dot{x} = (2gy)^{1/2}\cos\gamma$$

$$\dot{y} = (2gy)^{1/2}\sin\gamma$$

$$x(0) = y(0) = 0$$

where x is horizontal distance, y is vertical distance (positive downward), g is the acceleration due to gravity, and γ is path angle to the horizontal (see Fig. 2), find $\gamma(t)$ to minimize the time to reach $x = l$ with the constraint that $y \leq x\tan\theta + h$, with θ and h const.

This is a problem with a first-order state variable inequality constraint, since $S = y - x\tan\theta - h \leq 0$ does not contain the control variable γ, and $\dot{S} = (2gy)^{1/2}\sec\theta\sin(\gamma - \theta)$ does contain the control variable. On $S = 0$, $\dot{S} = 0$ implies that $\gamma = \theta$.

The solution to the unconstrained problem, $h/l \geq (2/\pi)\{1 - [(\pi/2) - \theta]\tan\theta\}$, is as follows:

$$\gamma(t) = \frac{\pi}{2} - \omega t \qquad \text{where } \omega = \left(\frac{\pi}{4}\frac{g}{l}\right)^{1/2}$$

$$\frac{x}{l} = \frac{2}{\pi}\left(\omega t - \frac{\sin 2\omega t}{2}\right)$$

$$\frac{y}{l} = \frac{2}{\pi}\sin^2\omega t$$

$$t_f = \left(\frac{\pi l}{g}\right)^{1/2} = \text{minimum final time}$$

$$\left.\begin{aligned} \lambda_x &= -\omega/g \\ \lambda_y &= -\frac{\omega}{g}\,\text{ctn}\,\omega t \end{aligned}\right\} \quad \text{where } dt_f = (\lambda_x\,\delta x + \lambda_y\,\delta y)_{t \leqslant t}$$

$$H = \lambda_x\dot{x} + \lambda_y\dot{y} = -1 = \text{variational Hamiltonian}$$

The solution to the constrained problem, $h/l < (2/\pi)\{1 - [(\pi/2) - \theta]\tan\theta\}$, is

$$\gamma(t) = \begin{cases} \frac{\pi}{2} - \omega_1 t & 0 \leq t \leq t_1 \\ \theta & t_1 \leq t \leq t_2 \\ \omega_2(t_f - t) & t_2 \leq t \leq t_f \end{cases}$$

where

$$\omega_1 = \left(\frac{g}{2}\,\frac{\theta - (\pi/2) + \text{ctn}\theta}{h\,\text{ctn}\theta}\right)^{1/2}$$

$$\omega_2 = \left(\frac{g}{2}\,\frac{\theta + \text{ctn}\theta}{l + h\,\text{ctn}\theta}\right)^{1/2}$$

$$t_1 = \frac{(\pi/2) - \theta}{\omega_1}$$

$$t_f - t_2 = \theta/2\omega_2$$

$$t_f = \left[\frac{2}{g}(l + h\,\text{ctn}\theta)(\theta + \text{ctn}\theta)\right]^{1/2} - \left[\frac{2h}{g}\,\text{ctn}\theta\left(\theta - \frac{\pi}{2} + \text{ctn}\theta\right)\right]^{1/2}$$

$= \text{minimum final time}$

$$\lambda_x(t_1-) - \lambda_x(t_1+) = -\mu_0\tan\theta$$

$$\lambda_y(t_1-) - \lambda_y(t_1+) = \mu_0$$

where $\mu_0 = (\text{ctn}\theta/g)(\omega_2 - \omega_1)$. Note that $\mu_0 \to 0$ and $t_1 \to t_2$ as $h/l \to (2/\pi)\{1 - [(\pi/2) - \theta]\tan\theta\}$:

$$H = \lambda_x\dot{x} + \lambda_y\dot{y} = -1 \qquad 0 \leq t \leq t_f$$

Figure 2 shows the solutions for $\tan\theta = \frac{1}{2}$ for several values of h/l.

6. Analytical Example with a Second-Order State Variable Inequality Constraint

Given**

$$\dot{v} = a$$

$$\dot{x} = v$$

$$v(0) = -v(1) = 1$$

$$x(0) = x(1) = 0$$

find $a(t)$ in $0 \leq t \leq 1$ to minimize

$$J = \frac{1}{2}\int_0^1 a^2 dt$$

with the constraint that $x(t) \leq l$. This is a problem with a second-order state variable inequality constraint, since $S = x - l$ and $\dot{S} = v$ do not explicitly contain the control variable $a(t)$, whereas $\ddot{S} = a(t)$ does explicitly contain the control variable.

** This example was suggested by John V. Breakwell of Lockheed Missiles and Space Company.

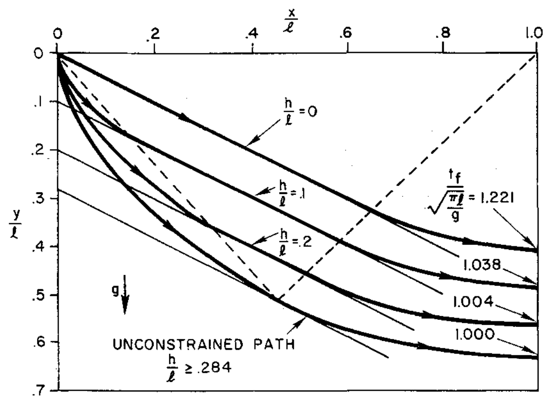

Fig. 2 Constrained Brachistochrone problem with $\tan\theta = \frac{1}{2}$.

The solution to the unconstrained problem, ($l \geq \frac{1}{4}$), is now obtained. Let $\dot{E} = \frac{1}{2}a^2$; $E(0) = 0$. Then $E(1)$ must be maximized. The Euler-Lagrange equations are

$$\begin{aligned} \dot{\lambda}_v &= -\lambda_x & \lambda_v &= -\lambda_x t + \text{const} \\ \dot{\lambda}_x &= 0 & \lambda_x &= \text{const} \\ \dot{\lambda}_E &= 0 & \lambda_E &= \text{const} = 1 \\ a &= -\lambda_v \end{aligned}$$

The solution is easily obtained as

$$\begin{aligned} a &= -2 \\ v &= 1 - 2t \\ x &= t(1-t) \rightarrow (x)_{\max} = \tfrac{1}{4} \\ \lambda_v &= -a = 2 \\ \lambda_x &= 0 \\ J &= 2 \\ H &= \lambda_x \dot{x} + \lambda_v \dot{v} + \lambda_E \dot{E} = -2 \end{aligned}$$

The solution with constraint, $\frac{1}{6} \leq l \leq \frac{1}{4}$, is obtained as follows:

$$a = \begin{cases} -8(1-3l) + 24(1-4l)t & 0 \leq t \leq \frac{1}{2} \\ -8(1-3l) + 24(1-4l)(1-t) & \frac{1}{2} < t < 1 \end{cases}$$

$$v = \begin{cases} 1 - 8(1-3l)t + 12(1-4l)t^2 & 0 \leq t \leq \frac{1}{2} \\ -1 + 8(1-3l)(1-t) - 12(1-4l)(1-t)^2 & \frac{1}{2} < t < 1 \end{cases}$$

$$x = \begin{cases} t - 4(1-3l)t^2 + 4(1-4l)t^3 & 0 \leq t \leq \frac{1}{2} \\ 1 - t - 4(1-3l)(1-t)^2 + 4(1-4l)(1-t)^3 & \frac{1}{2} < t < 1 \end{cases}$$

$$\lambda_v = -a \rightarrow \lambda_v(\tfrac{1}{2}-) - \lambda_v(\tfrac{1}{2}+) = 0 \ (\ddot{S} = 0 \text{ not used here})$$

$$\lambda_x = \begin{cases} 24(1-4l) & 0 \leq t \leq \frac{1}{2} \\ -24(1-4l) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Note that $\lambda_x(\frac{1}{2}-) - \lambda_x(\frac{1}{2}+) = 48(1-4l)$:

$$J = 2 + 6(1-4l)^2$$

$$H = -8(1-6l)^2$$

The solution with constraint, $0 < l \leq \frac{1}{6}$, is

$$a = \begin{cases} -\dfrac{2}{3l}\left(1 - \dfrac{t}{3l}\right) & 0 \leq t \leq 3l \\ 0 & 3l \leq t \leq 1 - 3l \\ -\dfrac{2}{3l}\left(1 - \dfrac{1-t}{3l}\right) & 1 - 3l \leq t \leq 1 \end{cases}$$

$$v = \begin{cases} \left(1 - \dfrac{t^2}{3l}\right) & 0 \leq t \leq 3l \\ 0 & 3l \leq t \leq 1 - 3l \\ -\left(1 - \dfrac{1-t}{3l}\right)^2 & 1 - 3l \leq t \leq 1 \end{cases}$$

$$x = \begin{cases} l\left[1 - \left(1 - \dfrac{t}{3l}\right)^3\right] & 0 \leq t \leq 3l \\ l & 3l \leq t \leq 1 - 3l \\ l\left[1 - \left(1 - \dfrac{1-t}{3l}\right)^3\right] & 1 - 3l \leq t \leq 1 \end{cases}$$

$$\lambda_v = \begin{cases} \dfrac{2}{3l}\left(1 - \dfrac{t}{3l}\right) & 0 \leq t \leq 3l \\ \dfrac{2}{3l}\left(1 - \dfrac{1-t}{3l}\right) & 3l \leq t \leq 1 \end{cases}$$

Note that $\lambda_v(3l-) - \lambda_v(3l+) = (4/3l^2)(\frac{1}{6} - l)$:

$$\lambda_x = \begin{cases} 2/9l^2 & 0 \leq t \leq 3l \\ -2/9l^2 & 3l \leq t \leq 1 \end{cases}$$

Note that $\lambda_x(3l-) - \lambda_x(3l+) = 4/9l^2$:

$$J = 4/9l$$

$$H = 0$$

Figure 3 shows the solutions for various values of l. The most interesting thing about these solutions is the fact that the optimal path touches the constraint boundary at only one point for a finite range of values of the constraint parameter ($\frac{1}{6} \leq l \leq \frac{1}{4}$), and only one of the influence functions λ_x is discontinuous. For $0 < l < \frac{1}{6}$, the path stays on the constraint boundary for a finite time, and both λ_v and λ_x are discontinuous. This behavior is typical of a second-order state variable inequality constraint.

Appendix A: Derivation of Necessary Conditions for an Extremal Solution with a qth Order State Variable Inequality Constraint

In Sec. 4, it was shown that the first q-1 derivatives of the state variable constraint function $S(\mathbf{x},t)$ must be zero at the entry corner of an $S = 0$ period, if $S^{(q)}[\mathbf{x}(t),\alpha(t),t]$ is the first derivative of S to be an explicit function of α. One thus has a vector of constraints at $[\mathbf{x}(t_1),t_1]$ of the same form as the $\mathbf{M}[\mathbf{x}(t_f),t_f] = 0$ constraints. They are written as

$$\mathbf{N}[\mathbf{x}(t_1),t_1] = \begin{bmatrix} S \\ S^{(1)} \\ \cdot \\ \cdot \\ S^{(q-1)} \end{bmatrix}_{t=t_1} = 0 \qquad \text{(A1)}$$

In a standard approach to the variational calculus problem defined in Sec. 2, with the added constraints (A1), the constraint

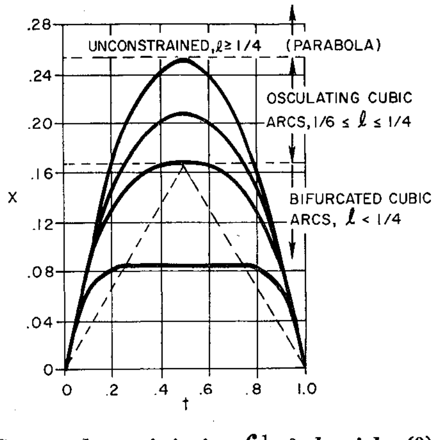

Fig. 3 Curves that minimize $\int_0^1 a^2\, dt$ with $x(0) = x(1) = v(0) = -v(1) = 1$, and $x \leq l$.

relations are adjoined to the performance index by means of Lagrange multipliers:

$$J = \phi + \boldsymbol{\nu}^T\mathbf{M} + \boldsymbol{\mu}^T\mathbf{N} + \int_{t_0}^{t_1-} + \int_{t_1+}^{t_2-} + \int_{t_2+}^{t_f} \boldsymbol{\lambda}^T(\mathbf{f} - \dot{\mathbf{x}})dt \quad \text{(A2)}$$

where $\boldsymbol{\nu}$ is a p-vector of Lagrange multiplier constants, $\boldsymbol{\mu}$ is a q-vector of Lagrange multiplier constants, and $\boldsymbol{\lambda}(t)$ is an n-vector of Lagrange multiplier functions. The integral has been written in three sections to allow for possible discontinuities at t_1 and t_2, the entry and exit corner times.

On an extremal solution, J must be stationary with respect to arbitrary small perturbations in the solution. This means that ϕ is stationary with respect to any small perturbations that satisfy the constraints (2.2, 2.3, 2.6, and A1). The differential of J is

$$dJ = \frac{\partial \phi}{\partial \mathbf{x}(t_f)} d\mathbf{x}_f + \frac{\partial \phi}{\partial t_f} dt_f + \boldsymbol{\nu}^T \left(\frac{\partial \mathbf{M}}{\partial \mathbf{x}(t_f)} d\mathbf{x}_f + \frac{\partial \mathbf{M}}{\partial t_f} dt_f\right) + \boldsymbol{\mu}^T \left(\frac{\partial \mathbf{N}}{\partial \mathbf{x}(t_1)} d\mathbf{x}_1 + \frac{\partial \mathbf{N}}{\partial t_1} dt_1\right) + \int_{t_0}^{t_1-} + \int_{t_1+}^{t_2-} + \int_{t_2+}^{t_f} \boldsymbol{\lambda}^T \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \delta\mathbf{x} + \frac{\partial \mathbf{f}}{\partial \alpha} \delta\alpha - \delta\dot{\mathbf{x}}\right) dt \quad \text{(A3)}$$

where $d\mathbf{x}_f = d\mathbf{x}(t_f)$, etc., and all partial derivatives are evaluated on the nominal (extremal) solution. Over each of the time intervals, the $\boldsymbol{\lambda}^T\delta\dot{\mathbf{x}}$ term first is integrated by parts. In the interval $t_1 < t < t_2$, (3.2) is used, with $S^{(q)}$ in place of C, to solve for $\delta\alpha$ in terms of $\delta\mathbf{x}$. One obtains

$$dJ = \frac{\partial \phi}{\partial \mathbf{x}(t_f)} d\mathbf{x}_f + \frac{\partial \phi}{\partial t_f} dt_f + \boldsymbol{\nu}^T \left(\frac{\partial \mathbf{M}}{\partial \mathbf{x}(t_f)} d\mathbf{x}_f + \frac{\partial \mathbf{M}}{\partial t_f} dt_f\right) + \boldsymbol{\mu}^T \left(\frac{\partial \mathbf{N}}{\partial \mathbf{x}(t_1)} d\mathbf{x}_1 + \frac{\partial \mathbf{N}}{\partial t_1} dt_1\right) - [(\boldsymbol{\lambda}^T\delta\mathbf{x})_{t=t_f} - (\boldsymbol{\lambda}^T\delta\mathbf{x})_{t=t_2+}] + \int_{t_2+}^{t_f} \left[\left(\dot{\boldsymbol{\lambda}}^T + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right) \delta\mathbf{x} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \alpha} \delta\alpha\right]dt - [(\boldsymbol{\lambda}^T\delta\mathbf{x})_{t=t_2-} - (\boldsymbol{\lambda}^T\delta\mathbf{x})_{t=t_1+}] + \int_{t_1+}^{t_2-} \left[\dot{\boldsymbol{\lambda}}^T + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \boldsymbol{\lambda}^T \left(\frac{\partial S^{(q)}}{\partial \alpha}\right)^{-1} \frac{\partial S^{(q)}}{\partial \mathbf{x}}\right] \delta\mathbf{x}dt - [(\boldsymbol{\lambda}^T\delta\mathbf{x})_{t=t_1-} - (\boldsymbol{\lambda}^T\delta\mathbf{x})_{t=t_0}] + \int_{t_0}^{t_1-} \left[\left(\dot{\boldsymbol{\lambda}}^T + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right) \delta\mathbf{x} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \alpha} \delta\alpha\right] dt \quad \text{(A4)}$$

For J to be stationary under arbitrary perturbations, the coefficients of $d\mathbf{x}_f$, dt_f, $d\mathbf{x}_1$, dt_1, $\delta\mathbf{x}(t)$, $\delta\alpha(t)$ must each be zero [$\delta\mathbf{x}(t_0) = 0$ by (2.4)]. To group terms, use

$$d\mathbf{x}_a = d[\mathbf{x}(t_a)] = \delta\mathbf{x}(t_a) + \dot{\mathbf{x}}(t_a)dt_a \quad \text{(A5)}$$

where $\mathbf{x}$ is evaluated at t_a, $\delta\mathbf{x}(t_a)$ is the change in $\mathbf{x}$ at the nominal t_a, and $\dot{\mathbf{x}}(t_a)dt_a$ is the change in $\mathbf{x}(t_a)$ due to the change in t_a. Using (A5) at t_f, t_2+, t_2-, t_1+, t_1-, (A4) is rearranged to obtain

$$dJ = \left[\frac{\partial \phi}{\partial \mathbf{x}(t_f)} + \boldsymbol{\nu}^T \frac{\partial \mathbf{M}}{\partial \mathbf{x}(t_f)} - \boldsymbol{\lambda}^T(t_f)\right]d\mathbf{x}_f + \left[\frac{\partial \phi}{\partial t_f} + \boldsymbol{\nu}^T \frac{\partial \mathbf{M}}{\partial t_f} + (\boldsymbol{\lambda}^T\dot{\mathbf{x}})_{t=t_f}\right] dt_f + [\boldsymbol{\lambda}^T(t_2+) - \boldsymbol{\lambda}^T(t_2-)]d\mathbf{x}_2 + [-(\boldsymbol{\lambda}^T\dot{\mathbf{x}})_{t=t_2+} + (\boldsymbol{\lambda}^T\dot{\mathbf{x}})_{t=t_2-}]dt_2 + \left[\boldsymbol{\mu}^T \frac{\partial \mathbf{N}}{\partial \mathbf{x}(t_1)} + \boldsymbol{\lambda}^T(t_1+) - \boldsymbol{\lambda}^T(t_1-)\right] d\mathbf{x}_1 + \left[\boldsymbol{\mu}^T \frac{\partial \mathbf{N}}{\partial t_1} - (\boldsymbol{\lambda}^T\dot{\mathbf{x}})_{t=t_1+} + (\boldsymbol{\lambda}^T\dot{\mathbf{x}})_{t=t_1-}\right] dt_1 + \int_{t_0}^{t_1-} + \int_{t_2+}^{t_f} \left[\left(\dot{\boldsymbol{\lambda}}^T + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right) \delta\mathbf{x} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \alpha} \delta\alpha\right] dt + \int_{t_1+}^{t_2-} \left\{\dot{\boldsymbol{\lambda}}^T + \boldsymbol{\lambda}^T \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \frac{\partial \mathbf{f}}{\partial \alpha} \left(\frac{\partial S^{(q)}}{\partial \alpha}\right)^{-1} \frac{\partial S^{(q)}}{\partial \mathbf{x}}\right]\right\} \delta\mathbf{x}dt \quad \text{(A6)}$$

For stationary J, then, one must have

$$\left.\begin{aligned} \boldsymbol{\lambda}^T(t_f) &= \frac{\partial \phi}{\partial \mathbf{x}(t_f)} + \boldsymbol{\nu}^T \frac{\partial \mathbf{M}}{\partial \mathbf{x}(t_f)} \\ (\boldsymbol{\lambda}^T\dot{\mathbf{x}})_{t=t_f} &= -\left(\frac{\partial \phi}{\partial t_f} + \boldsymbol{\nu}^T \frac{\partial \mathbf{M}}{\partial t_f}\right) \end{aligned}\right\} \quad \text{(A7)}$$

$$\left.\begin{aligned} \boldsymbol{\lambda}(t_2-) &= \boldsymbol{\lambda}^T(t_2+) \\ (\boldsymbol{\lambda}^T\dot{\mathbf{x}})_{t=t_2-} &= (\boldsymbol{\lambda}^T\dot{\mathbf{x}})_{t=t_2+} \end{aligned}\right\} \quad \text{(A8)}$$

$$\left.\begin{aligned} \boldsymbol{\lambda}^T(t_1-) &= \boldsymbol{\lambda}^T(t_1+) + \boldsymbol{\mu}^T \frac{\partial N}{\partial \mathbf{x}(t_1)} \\ (\boldsymbol{\lambda}^T\dot{\mathbf{x}})_{t=t_1-} &= (\boldsymbol{\lambda}^T\dot{\mathbf{x}})_{t=t_1+} - \boldsymbol{\mu}^T \frac{\partial \mathbf{N}}{\partial t_1} \end{aligned}\right\} \quad \text{(A9)}$$

$$\left.\begin{aligned} \dot{\boldsymbol{\lambda}} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^T \boldsymbol{\lambda} &= 0 \\ \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \alpha} &= 0 \end{aligned}\right\} \quad S < 0 \quad \text{(A10)}$$

$$\left.\begin{aligned} \dot{\boldsymbol{\lambda}} + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{f}}{\partial \alpha} \frac{\partial S^{(q)}}{\partial \alpha}\right)^{-1} \frac{\partial S^{(q)}}{\partial \mathbf{x}}\right]^T \boldsymbol{\lambda} &= 0 \\ S^{(q)}[\mathbf{x},\alpha,t] &= 0 \end{aligned}\right\} \quad S = 0 \quad \text{(A11)}$$

By identifying

$$\boldsymbol{\lambda}^T(t_1+) = \frac{\partial \phi}{\partial \mathbf{x}(t_1)} \qquad (\boldsymbol{\lambda}^T\dot{\mathbf{x}})_{t=t_1+} = -\frac{\partial \phi}{\partial t_1} \quad \text{(A12)}$$

it is seen that (A9) is completely analogous to the terminal boundary conditions (A7).

With a control variable inequality constraint $C(\mathbf{x},\alpha,t) \leq 0$ (which is the special case $q = 0$), there is no $\mathbf{N}$ vector and hence no discontinuity in $\boldsymbol{\lambda}$ at either t_1 or t_2.

One may use the idea of an additional (scalar) constraint $N[\mathbf{x}(t_1),t_1] = 0$ to allow certain discontinuities in the path. First, suppose that $\mathbf{f}(\mathbf{x},\alpha,t)$ has a finite discontinuity on some surface $N(\mathbf{x},t) = 0$. Then, suppose that the extremal solution reaches this surface at time t_1. One can think of this as a constraint

$$N[\mathbf{x}(t_1),t_1] = 0 \quad \text{(A13)}$$

The analysis of this appendix may then be followed through to obtain, in particular, (A9). Since $\mathbf{f}(t_1+) \neq \mathbf{f}(t_1-)$ has been assumed, one can see immediately that the second equation of (A9) requires a discontinuity in $\boldsymbol{\lambda}$ at t_1.[††] With continuous $\mathbf{f}$, $\mu = 0$ is obtained as the only possible solution; t_1, in fact, is then of no special significance. One has $n + 2$ equations in (A9) and (A13) for the n jumps in the influence functions, for the time t_1, and for the multiplier μ. Note that Breakwell obtained essentially the same results in Ref. 3.

The allowable discontinuity can be pushed one step further by assuming that $\mathbf{f}$ becomes infinite on a surface $N(\mathbf{x},t) = 0$, but that the resulting discontinuity $\Delta\mathbf{x}$ is a known function, say, $\mathbf{A}(\mathbf{x},t)$. It again is assumed that the extremal solution reaches the surface $N = 0$ at time t_1. One again invokes the constraint

$$N[\mathbf{x}(t_1),t_1] = 0 \quad \text{(A14)}$$

(A9) is no longer valid, however, because (A5) and (A6) do not adequately describe a perturbed solution. Before, $d\mathbf{x}(t_a)$

[††] Unless $\boldsymbol{\lambda}$ has a zero component for each nonzero component of $\mathbf{f}(t_1-) - \mathbf{f}(t_1+)$.

was assumed the same, whether viewed from t_a- or from t_a+. Now, if changes $dt_1, d\mathbf{x}(t_1-)$ occur, one obtains

$$d\mathbf{x}(t_1+) = d\mathbf{x}(t_1-) + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\right)_{t_1-} d\mathbf{x}(t_1-) + \left(\frac{\partial A}{\partial t}\right)_{t_1-} dt_1$$

$$= d\mathbf{x}(t_1-)\left[\mathbf{I} + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\right)_{t_1-}\right] + \left(\frac{\partial \mathbf{A}}{\partial t}\right)_{t_1-} dt_1 \qquad \text{(A15)}$$

Rewriting (A6) with dx_1+ and dx_1- distinct and then substituting for dx_1+ from (A14), one finds that (A9) is replaced by

$$\left.\begin{aligned} \lambda^T(t_1-) &= \lambda^T(t_1+)\left[\mathbf{I} + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\right)_{t_1-}\right] + \mu\left(\frac{\partial N}{\partial \mathbf{x}}\right)_{t_1-} \\ \lambda^T(t_1-)\dot{\mathbf{x}}(t_1-) &= \lambda^T(t_1+)\left[\dot{\mathbf{x}}(t_1+) - \left(\frac{\partial \mathbf{A}}{\partial t}\right)_{t_1-}\right] - \mu\left(\frac{\partial N}{\partial t}\right)_{t_1-} \end{aligned}\right\} \qquad \text{(A16)}$$

Note that no smallness requirement was placed on the discontinuity $\Delta\mathbf{x}$, but the changes $d\mathbf{x}(t_1-)$ and dt_1 and the resulting $d\mathbf{x}(t_1+)$ on perturbed paths must be of first order only.

Appendix B: Nonuniqueness of Influence Functions on a State Variable Inequality Constraint Boundary

On a state variable inequality constraint boundary, one has

$$\dot{\lambda}^T + \lambda^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \lambda^T \frac{\partial \mathbf{f}}{\partial \alpha}\left(\frac{\partial S^{(q)}}{\partial \alpha}\right)^{-1} \frac{\partial S^{(q)}}{\partial \mathbf{x}} = 0 \qquad \text{(B1)}$$

If one adds the function $b(\partial S^{(q-1)}/\partial \mathbf{x})$ to λ^T each place it appears in (B1), one obtains

$$\frac{d}{dt}\left(\lambda^T + b\frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\right) + \left(\lambda^T + b\frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\right)\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \left(\lambda^T + b\frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\right)\frac{\partial \mathbf{f}}{\partial \alpha}\left(\frac{\partial S^{(q)}}{\partial \alpha}\right)^{-1}\frac{\partial S^{(q)}}{\partial \mathbf{x}} \qquad \text{(B2)}$$

Subtracting (B1) from (B2), one obtains

$$\frac{d}{dt}\left(d\frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\right) + b\frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - b\frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\frac{\partial \mathbf{f}}{\partial \alpha}\left(\frac{\partial S^{(q)}}{\partial \alpha}\right)^{-1}\frac{\partial S^{(q)}}{\partial \mathbf{x}} \qquad \text{(B3)}$$

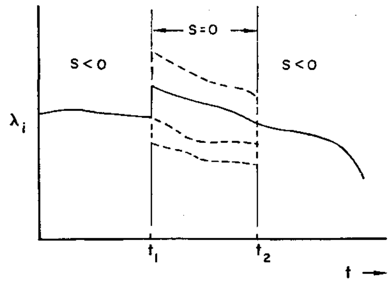

Fig. 4 Typical influence function history with a state variable inequality constraint.

But

$$\frac{\partial S^{(q)}}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\mathbf{f} + \frac{\partial S^{(q-1)}}{\partial t}\right] = \frac{\partial S^{(q-1)}}{\partial \mathbf{x}}\frac{\partial \mathbf{f}}{\partial \alpha} \qquad \text{(B4)}$$

since $S^{(q-1)}$ is not an explicit function of α. The last term of (B3) is thus $-b(\partial S^{(q)}/\partial \mathbf{x})$. By expanding the first term of (B3) and combining terms, one finds that (B3) reduces to

$$\frac{db}{dt}\frac{\partial S^{(q-1)}}{\partial \mathbf{x}} \qquad \text{(B5)}$$

If b is constant, one has that (B2) is zero. Adding $b(\partial S^{(q-1)}/\partial \mathbf{x})$ to λ^T in the interval on the constraint boundary thus leaves the differential equation (B1) unchanged.

Figure 4 shows possible histories of a component of λ across an interval where $S = 0$. The solid line is the choice made in this paper, namely, $b = -b' = 0$, which makes λ continuous at exit corners like $t = t_2$.

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